Dynamic modeling and control properties of underactuated MRAVs

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Room Calypso A – Divani Corfu Palace

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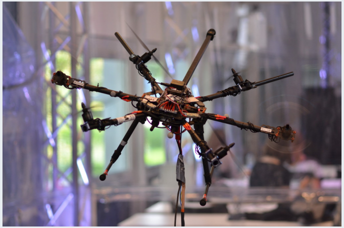
Multi-Rotor Aerial Vehicles (MRAVs)

Under-actuated MRAVs (u-MRAVs)

- Controllable DoFs: 4 (3 position, 1 rotation),
- Wide spread; commercially viable since 2010,
- Balance between capability and simplicity; differentially flat system.

Fully-actuated MRAVs (f-MRAVs)



- Controllable DoFs: 6 (3 position, 3 rotation),
- Mainly subject of research: impedance control, force application,
- Wide range of designs with both fixed and tiltable propellers.

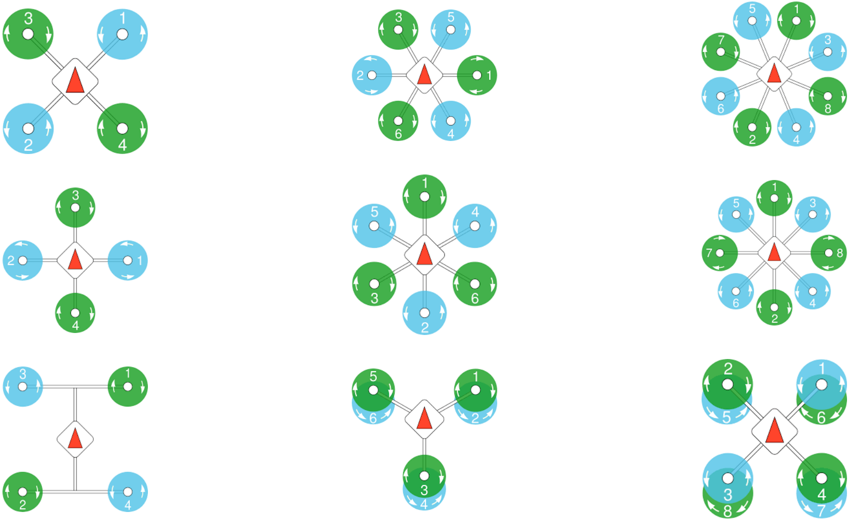
u-MRAVs

Common properties

- ≥ 4 fixed-pitch and non-tiltable propellers,
- Individually-controlled propeller speed,
- Propeller axes perpendicular to fuselage plane,
- Widely supported in simulation and control frameworks.

Common configurations

- Quadrotor (cross, plus),
- Hexarotor (cross, plus, coaxial Y),
- Octarotor (cross, plus, coaxial X),
- Suitable for agile tasks in cluttered environments.



3 1 2 4
3 5 2 1 6 4
5 1 7 3 6 8 2 4
3 2 1 4
1 5 4 3 6 2
1 5 3 7 8 6 4 2
3 1 2 4
5 1 6 2 3 4
2 1 5 6 3 4 8 7

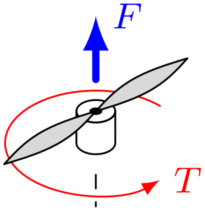
Single propeller model

Propeller thrust model (simplified)

Thrust is produced due to propellers' lift

$$F \approx k\omega^2$$

- F: thrust force [N],
- k: linear coefficient [$\mathrm{N\,s^2\,rad^{-2}}$],
- ω: propeller rate [$\mathrm{rad\,s^{-1}}$].



Motor dynamics (closed loop)

$$\dot{\omega} = -\frac{1}{\tau_m}(\omega - \omega_d)$$

- τ_m: time constant, $\approx 30\,\mathrm{ms}$,
- ω: propeller rate [$\mathrm{rad\,s^{-1}}$],
- ω_d: desired propeller rate [$\mathrm{rad\,s^{-1}}$].

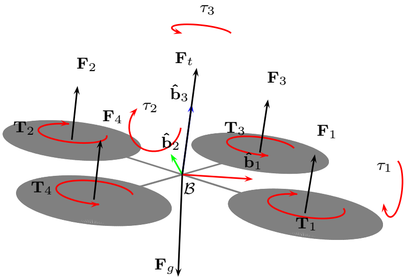
Propeller torque model (simplified)

Torque is produced due to propellers' drag

$$T \approx c_t F$$

- F: thrust force [N],
- c_t: linear torque coefficient [m].

u-MRAV (cross) dynamics model



- Right-handed body-fixed coordinate frame $\mathcal{B} = \{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$
- Propellers produce thrust forces: $F_1 = \|\mathbf{F}_1\|$, $F_2 = \|\mathbf{F}_2\|$, $F_3 = \|\mathbf{F}_3\|$, $F_4 = \|\mathbf{F}_4\|$
- Gravity force acts on the center of mass,
- Propellers produce torques: $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$,
- Motor speeds $\omega_{1\text{-}4}$ generate forces $F_{1\text{-}4}$, producing torques $\tau_{1\text{-}3}$ and total thrust F_t.

Force-Torque allocation

$$\begin{bmatrix} F_t \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ -d/\sqrt{2} & d/\sqrt{2} & d/\sqrt{2} & -d/\sqrt{2} \\ -d/\sqrt{2} & d/\sqrt{2} & -d/\sqrt{2} & d/\sqrt{2} \\ -c_t & -c_t & c_t & c_t \end{bmatrix}}_{\mathbf{\Gamma},\ \text{Allocation matrix, } \det(\mathbf{\Gamma})\neq 0 \text{ when} \{d, c_t\} \neq 0} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

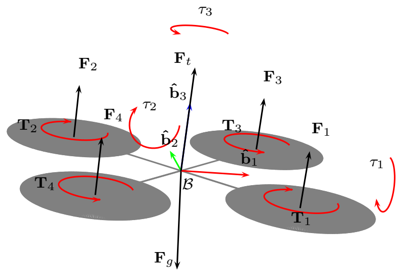
u-MRAV (cross) dynamics model

Control input mapping

From force and torques to per-motor angular speed:

$$\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\\\omega_4\end{bmatrix} = \sqrt{\boldsymbol{\Gamma}^{-1}\begin{bmatrix}F_t\\\tau_1\\\tau_2\\\tau_3\end{bmatrix}k^{-1}}$$

- The input mapping is a static transformation
- There exist $\boldsymbol{\Gamma}$ for each common multi-rotor frame configuration
- Parameters k and c_t need to be identified,
- For more accurate model, the transient $\omega(s)/\omega_d(s) = 1/\tau_m(s+1)$ needs to be considered, due to motor and propeller inertia



Rotation matrix

The rotation matrix $\mathbf{R} \in \mathrm{SO}(3)$ maps body-frame vectors to the world frame:

$$\mathbf{v}^{\mathcal{W}} = \mathbf{R}\,\mathbf{v}^{\mathcal{B}}$$

Critical for closed-loop control.

u-MRAV dynamics

2nd Newton's law – translational

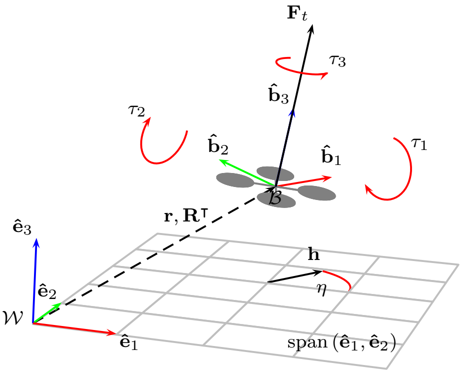
$$\ddot{\mathbf{r}}^{\mathcal{W}} = \frac{1}{m}\mathbf{R}\begin{bmatrix}0\\0\\F_t\end{bmatrix}^{\mathcal{B}} + \mathbf{g}^{\mathcal{W}}$$

- $\mathbf{r}^{\mathcal{W}}$: position vector [m],
- $\mathbf{g}$: gravity vector $[0, 0, -9.8]^\top$ $[\mathrm{m\,s^{-2}}]$.

Euler's equation of motion – rotational

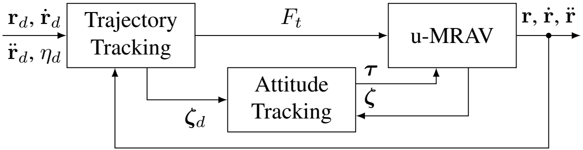
$$\boldsymbol{\tau} = \underbrace{\boldsymbol{J}\dot{\boldsymbol{\zeta}}}_{\text{rot. motion}} + \underbrace{\boldsymbol{\zeta} \times \boldsymbol{J}\boldsymbol{\zeta}}_{\text{precession}}$$

- m: mass [kg]; $\boldsymbol{\zeta}$: angular velocity,
- $\boldsymbol{J}$: moment of inertia $\in \mathbb{R}^{3\times 3}$ $[\mathrm{kg\,m^2}]$.



u-MRAV control history

- Near-hover **linear controllers** (PD, LQR)
- **Nonlinear controllers:** dynamic feedback linearization, backstepping and sliding model controllers, geometric controllers
- **Optimal controllers:** RL, MPC



Control law

$$F_t = m\left(\ddot{\mathbf{r}}_d + \mathbf{K}_p\mathbf{e}_p + \mathbf{K}_v\mathbf{e}_v + \mathbf{g}\right)\mathbf{R}\mathbf{e}_3$$

$$\boldsymbol{\tau} = -\mathbf{K}_R\mathbf{e}_R - \mathbf{K}_\zeta\mathbf{e}_\zeta + \boldsymbol{\zeta}\times\mathbf{J}\boldsymbol{\zeta} - \mathbf{J}\left(\hat{\boldsymbol{\zeta}}\mathbf{R}^\top\mathbf{R}_d\boldsymbol{\zeta}_d - \mathbf{R}^\top\mathbf{R}_d\dot{\boldsymbol{\zeta}}_d\right)$$

- $\mathbf{e}_p$, $\mathbf{e}_v$: position and velocity errors,
- $\mathbf{K}_p$, $\mathbf{K}_v$: position and velocity gain matrices,
- $\mathbf{K}_R$, $\mathbf{K}_\zeta$: rotation and angular velocity gain matrices.
- $\mathbf{e}_R$, $\mathbf{e}_\zeta$: rotation and angular velocity errors
- $\mathbf{e}_\zeta = \boldsymbol{\zeta} - \mathbf{R}^\top\mathbf{R}_d\boldsymbol{\zeta}_d$
- $\mathbf{e}_R = {}^1\!/_2(\mathbf{R}_d^\top\mathbf{R} - \mathbf{R}^\top\mathbf{R}_d)^\vee$, $(\cdot)^\vee : SO(3) \to \mathbb{R}^3$

u-MRAV control: optimal regulators

Model predictive control

$$\begin{aligned}\underset{\mathbf{u},\,\mathbf{x}}{\text{minimize}}\quad & \|\mathbf{x}_N - \mathbf{x}_{N_d}\| + \sum_{k=0}^{N-1}\left(\|\mathbf{x}_k - \mathbf{x}_{k_d}\| + \|\mathbf{u}_k - \mathbf{u}_{k_d}\|\right)\\ \text{s.t.}\quad & \text{current state,}\\ & \text{motion model,}\\ & \text{actuation and state bounds.}\end{aligned}$$

- At a **fixed frequency**, solve an optimal control problem.
- **Execute** first action and **repeat**.

Reinforcement learning

- Use a simulator to train a policy
  - **Input:** state; **Output:** motor speed, desired force/torque, or attitude setpoint
- For sim-to-real transfer use **domain randomization**
  - Sample different inertia and physical parameters
  - Potentially add (motor) delays

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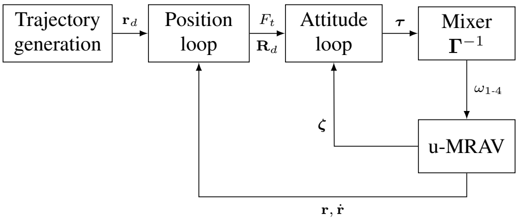
u-MRAV control: cascaded architecture

Time-scale separation

Translational and rotational dynamics are coupled through $\mathbf{R}$, but evolve on **different time scales**. The cascade exploits this: an outer (slow) position loop produces a thrust magnitude F_t and a desired attitude $\mathbf{R}_d$; an inner (fast) attitude loop tracks $\mathbf{R}_d$ via body torques $\boldsymbol{\tau}$.



- Outer loop (**position**): ~ 50 Hz–100 Hz, gain-limited by sensor latency,
- Inner loop (**attitude**): ~ 500 Hz–1 kHz, runs on the autopilot MCU,
- Mixer: instantaneous algebraic inversion of $\boldsymbol{\Gamma}$,
- **Bandwidth ratio** ≥ 5 keeps the cascade well-posed.

u-MRAV control: differential flatness

A foundational structural property

A system is **differentially flat** if there exist outputs $\mathbf{y}$ such that all states $\mathbf{x}$ and inputs $\mathbf{u}$ can be written as algebraic functions of $\mathbf{y}$ and a finite number of its time derivatives, i.e. **no integration is required**.

Flat outputs of the quadrotor

Mellinger & Kumar (2011) showed that

$$\mathbf{y} = \begin{bmatrix} \mathbf{r}^{\mathcal{W}} \\ \psi \end{bmatrix} \in \mathbb{R}^4$$

are flat outputs of the u-MRAV.

- $\ddot{\mathbf{r}}_d \Rightarrow F_t$ and the desired thrust direction $\hat{\mathbf{b}}_3^d$,
- $\dddot{\mathbf{r}}_d \Rightarrow$ desired body rates $\boldsymbol{\zeta}_d$,
- $\ddddot{\mathbf{r}}_d$ (*snap*) $\Rightarrow$ desired body accelerations $\dot{\boldsymbol{\zeta}}_d$,
- $\psi_d, \dot{\psi}_d \Rightarrow$ heading.

Why it matters

- **Planning** can be done in $\mathbb{R}^4$ (flat space) instead of full state space,
- Any smooth trajectory in $\mathbf{y}$ is dynamically feasible (up to actuator limits),
- Enables **feedforward** reference signals for the inner controllers,
- Holds **also under linear rotor drag** Faessler *et al.*, 2018).

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u-MRAV control: nonlinear control

Dynamic feedback linearization

The map $\mathbf{u} \mapsto \mathbf{r}$ has a relative degree 4 on each axis. By appending an integrator on F_t and choosing

$$\ddddot{\mathbf{r}} = \mathbf{v}, \qquad \mathbf{v} = \ddddot{\mathbf{r}}_d + \mathbf{K}_3\,\dddot{\mathbf{e}} + \mathbf{K}_2\ddot{\mathbf{e}} + \mathbf{K}_1\dot{\mathbf{e}} + \mathbf{K}_0\mathbf{e}$$

the closed loop becomes **linear and decoupled** along each Cartesian axis.

- Exact tracking of any sufficiently smooth trajectory,
- Sensitive to **model mismatch** (mass, drag),
- Requires snap in the reference.

Backstepping

Recursive Lyapunov design through the cascade $\mathbf{r} \to \dot{\mathbf{r}} \to \mathbf{R} \to \zeta$:

1. Define $V_1 = \frac{1}{2}\|\mathbf{e}_p\|^2$, pick virtual input $\dot{\mathbf{r}}^\star$,
2. Define $V_2 = V_1 + \frac{1}{2}\|\mathbf{e}_v\|^2$, pick $\mathbf{R}^\star$, F_t,
3. Continue until the actual input $\boldsymbol{\tau}$ appears,
4. Choose $\boldsymbol{\tau}$ so that $\dot{V}_n \leq 0$.

This approach

- Guarantees **global asymptotic stability** when Euler-angle singularities are avoided,
- Is easy to extend with integral / adaptive terms.

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Thank you

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